

# Asymptotic Freedom, Dimensional Transmutation, and an Infra-red Conformal Fixed Point for the $\delta$ -Function Potential in 1-dimensional Relativistic Quantum Mechanics

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## Abstract

We consider the Schrödinger equation for a relativistic point particle in an external 1-dimensional  $\delta$ -function potential. Using dimensional regularization, we investigate both bound and scattering states, and we obtain results that are consistent with the abstract mathematical theory of self-adjoint extensions of the pseudo-differential operator  $H = \sqrt{p^2 + m^2}$ . Interestingly, this relatively simple system is asymptotically free. In the massless limit, it undergoes dimensional transmutation and it possesses an infra-red conformal fixed point. Thus it can be used to illustrate non-trivial concepts of quantum field theory in the simpler framework of relativistic quantum mechanics.

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# 1 Introduction

The unification of quantum physics and special relativity is achieved in the framework of relativistic quantum field theories. In particular, in the standard model of particle physics elementary particles are very successfully described as quantized wave excitations of the corresponding quantum fields. As such, they have qualitatively different properties than the point particles of Newtonian mechanics or quantum mechanics. In particular, while the position of a quantum mechanical point particle is in general uncertain, quantized waves do not even have a conceptually well-defined position in space. Unlike in quantum mechanics, in local quantum field theory a “particle” is a non-local object [1–5]. It is well-known that a unification of point particle mechanics and special relativity is problematical, even at the classical level. In particular, Currie, Jordan, and Sudarshan proved that two point particles cannot interact in such a way that the principles of special relativity are respected, i.e. that the system provides a representation of the Poincaré algebra [6]. Leutwyler has generalized this result to an arbitrary number of particles [7]. His non-interaction theorem states that classical relativistic point particles are necessarily free, as a consequence of Poincaré invariance. The only exception are two particles in one spatial dimension confined to each other by a linearly rising potential. In one dimension, the corresponding confining string has no other degrees of freedom than the positions of its endpoints, which are represented by the two point particles. While strings can interact relativistically in higher dimensions, according to Leutwyler’s non-interaction theorem, point particles can not. Hence, it is not surprising that particle physics is based on quantum field theory rather than on relativistic point particle quantum mechanics. It should also be noted that, by including the interaction in the momentum and not in the boost operator, interesting relativistic systems with a fixed number of interacting particles have been constructed and investigated in detail [8–10]. However, in this case, the coordinates and momenta of the particles do not obey canonical commutation relations, and thus do not describe ordinary point particles.

When studying fundamental physics, it is a big step to proceed from non-relativistic quantum mechanics to relativistic quantum field theory. Not only for pedagogical reasons, it is interesting to ask whether non-trivial systems of relativistic quantum mechanics exist. Even free quantum mechanical relativistic point particles have some interesting properties [11–14]. Minimal position-velocity wave packets of such particles spread in such a way that probability leaks out of the light-cone. While such a quantum mechanical violation of causality does not happen in relativistic quantum field theories, it would arise in a hypothetical world of relativistic point particles [15–24]. While in quantum field theory a local Hamiltonian gives rise to non-local field excitations that manifest themselves as “particles”, in relativistic quantum mechanics local point particles of mass  $m$  follow the dynamics of the non-local Hamiltonian  $H = \sqrt{p^2 + m^2}$ . According to Leutwyler’s non-interaction theorem, one cannot add a potential to this Hamiltonian without violating the princi-

ples of relativity theory, already at the classical level. This is not surprising, because a potential would describe instantaneous interactions at a distance, mediated with infinite speed. The only exception are singular contact interactions, which are not excluded by the classical non-interaction theorem. Hence, there might be a quantum loop-hole in the theorem, which would be worth exploring, at least for pedagogical reasons, trying to bridge the large gap between non-relativistic quantum mechanics and relativistic quantum field theory in studying fundamental physics.

In non-relativistic quantum mechanics, contact interactions have been studied in great detail [25–34], which has been used to illustrate some non-trivial concepts of quantum field theories in the simpler context of non-relativistic quantum mechanics. In this paper we endow the Hamiltonian  $H = \sqrt{p^2 + m^2}$  for a single free relativistic point particle in one spatial dimension with a contact interaction potential  $\lambda\delta(x)$ . We can imagine that such a potential is generated by a second particle of infinite mass. Once this case is fully understood, as a next step one can then consider two relativistic particles of finite mass, and ask whether a contact interaction leads to a non-trivial representation of the Poincaré group, thus providing a quantum mechanical loop-hole in the classical non-interaction theorem. In this paper, we do not yet address that question and limit ourselves to a single particle in the external 1-dimensional  $\delta$ -function potential. Remarkably, already this relatively simple problem provides interesting insights into some qualitative differences between relativistic and non-relativistic quantum mechanics. While the simple  $\delta$ -function potential provides a standard textbook problem, a non-relativistic particle moving in one spatial dimension allows more general contact interactions. It can actually distinguish a 4-parameter family of such interactions. This follows from the theory of self-adjoint extensions [35, 36] of the local free-particle kinetic energy Hamiltonian  $H = \frac{p^2}{2m}$  [29, 37–41]. There is a 4-parameter family of self-adjoint extensions characterized by the boundary condition for the wave function at the contact point

$$\begin{pmatrix} \Psi(\varepsilon) \\ \partial_x \Psi(\varepsilon) \end{pmatrix} = \exp(i\theta) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Psi(-\varepsilon) \\ \partial_x \Psi(-\varepsilon) \end{pmatrix}. \quad (1.1)$$

Here  $\varepsilon \rightarrow 0$ ,  $a, b, c, d \in \mathbb{R}$  with  $ad - bc = 1$ , and  $\theta \in ]-\frac{\pi}{2}, \frac{\pi}{2}]$ . The five parameters  $a, b, c, d, \theta$  with the constraint  $ad - bc = 1$  provide a 4-parameter family of self-adjoint extensions of the non-relativistic free-particle Hamiltonian, and thus a 4-parameter family of quantum mechanical contact interactions. The standard contact interaction potential  $\lambda\delta(x)$  just corresponds to  $a = d = 1$ ,  $b = 0$ ,  $c = 2m\lambda$ , and  $\theta = 0$ . The most general contact interaction does not respect parity symmetry, which requires  $a = d$  and  $\theta = 0$ . Still, in the non-relativistic case, this leaves a 2-parameter family of parity-invariant contact interactions. A free particle with a generalized energy-momentum dispersion relation  $H = \sum_{n=0}^N c_n p^n$  even allows an  $N^2$ -parameter family of self-adjoint extensions. For very high momenta  $p$ , the energy of such a particle increases as  $p^N$ , which for  $N > 2$  allows the resolution of further details of a contact point than for the standard non-relativistic dispersion relation with  $N = 2$ . If one thinks of the relativistic energy-momentum dispersion relation  $H = \sqrt{p^2 + m^2}$  as a

power series expansion in  $p^2$  with  $N \rightarrow \infty$ , in the relativistic case one might perhaps expect an infinite number of self-adjoint extension parameters, and thus an infinite variety of contact interactions, e.g. represented by the  $\delta$ -function potential and all its derivatives. However, the opposite is true. At large momentum  $p$ , the relativistic energy  $\sqrt{p^2 + m^2}$  only increases as  $|p|$ , which provides less short-distance resolution than the non-relativistic  $p^2$ . Indeed, there is just a 1-parameter family of self-adjoint extensions of the relativistic free-particle Hamiltonian  $H = \sqrt{p^2 + m^2}$ , which can be characterized by the parameter  $\lambda$  in the contact interaction potential  $\lambda\delta(x)$ . This follows from the self-adjoint extension theory of so-called pseudo-differential operators, which includes the non-local Hamiltonian  $H = \sqrt{p^2 + m^2}$  [42]. This theory also predicts that in higher dimensions, relativistic point particles are completely unaffected by contact interactions and thus remain free. This is again in contrast to the non-relativistic case, in which there is a 1-parameter family of contact interactions both in two and in three spatial dimensions [29].

As a result of Leutwyler's non-interaction theorem as well as of the theory of self-adjoint extensions of the pseudo-differential operator  $H = \sqrt{p^2 + m^2}$ , relativistic quantum mechanics is a rather narrow subject. In particular, for a single particle one is limited to the simple  $\delta$ -function or to a linear confining potential. In this context, it is important to point out that the Klein-Gordon and Dirac equations do not belong to relativistic quantum mechanics, but to quantum field theory. In particular, it is well-known that these equations do not allow a consistent single-particle interpretation, because they address the physics of both particles and anti-particles. The relativistic point particle Hamiltonian  $H = \sqrt{p^2 + m^2}$ , on the other hand, is concerned just with particles. The problem of the relativistic  $\delta$ -function potential has already been investigated in the mathematical literature as an application of the theory of self-adjoint extensions of pseudo-differential operators [42]. Here we address the problem using more traditional tools of theoretical physics. Unlike in the non-relativistic case, the relativistic  $\delta$ -function potential gives rise to ultra-violet divergences which we regularize and renormalize using dimensional regularization [43–46]. It is reassuring that the results that we obtain are indeed consistent with those obtained by the self-adjoint extension theory of Ref.[42]. Here we study the system in great detail, and address various interesting physics questions, including strong bound states with a binding energy that exceeds the rest mass of the bound particle. Remarkably, this relatively simple quantum mechanical model shares several non-trivial features with relativistic quantum field theories. In particular, just like Quantum Chromodynamics (QCD) [47], it is asymptotically free [48, 49].

In two spatial dimensions, a non-relativistic  $\delta$ -function potential must also be renormalized [25–34]. While this system is classically scale invariant, at the quantum level it dynamically generates a bound state via dimensional transmutation, and it has scattering states which display asymptotic freedom. Hence, it can be used to illustrate these non-trivial features, which are usually encountered in quantum field theory, in the framework of non-relativistic quantum mechanics. How-

ever, this theory can not be obtained as the non-relativistic limit of a relativistic theory. In this paper, we show that asymptotic freedom and dimensional transmutation already arise in 1-dimensional relativistic point particle quantum mechanics with a  $\delta$ -function potential. Furthermore, in the massless limit the system is scale-invariant, at least at the classical level. However, just like in QCD, scale invariance is anomalously broken at the quantum level. The system then undergoes dimensional transmutation and generates a mass scale non-perturbatively. Unlike QCD, in the massless limit the relativistic quantum mechanical model even has a free infra-red conformal fixed point. Although actual elementary particles are quantized waves rather than point-like objects, addressing these topics in relativistic point particle quantum mechanics makes them more easily accessible than just studying them in the standard context of relativistic quantum field theories.

The rest of this paper is organized as follows. In section 2, we consider the bound state problem and use the bound state energy to define a renormalization condition. In section 3, we derive the relativistic probability current density and show explicitly that it is conserved. In section 4, we address the scattering states and we show that the energy-dependent running coupling constant is finite after renormalization. Reflection and transmission amplitudes, as well as the scattering phase shift, the scattering length, and the effective range are derived in section 5. In section 6, we investigate the energy-dependence of the running coupling constant and its  $\beta$ -function, and we show that the theory is asymptotically free. In section 7, we study ultra-strong bound states and the corresponding scattering states. Section 8 analyzes the massless limit, in which the system undergoes dimensional transmutation, and develops an infra-red conformal fixed point. Finally, section 9 contains our conclusions.

## 2 Dimensional Regularization and Renormalization of a Bound State

Let us consider the relativistic time-independent Schrödinger equation

$$\sqrt{p^2 + m^2}\Psi(x) + \lambda\delta(x)\Psi(x) = E\Psi(x). \quad (2.1)$$

In momentum space

$$\Psi(x) = \frac{1}{2\pi} \int dp \, \tilde{\Psi}(p) \exp(ipx), \quad \Psi(0) = \frac{1}{2\pi} \int dp \, \tilde{\Psi}(p), \quad (2.2)$$

and the Schrödinger equation takes the form

$$\sqrt{p^2 + m^2} \tilde{\Psi}(p) + \frac{\lambda}{2\pi} \int dp' \tilde{\Psi}(p') = E\tilde{\Psi}(p), \quad (2.3)$$

such that for a bound state

$$\tilde{\Psi}_B(p) = \frac{\lambda \Psi_B(0)}{E_B - \sqrt{p^2 + m^2}}. \quad (2.4)$$

Integrating this equation over all momenta, we obtain the gap equation

$$\begin{aligned} \Psi_B(0) &= \frac{1}{2\pi} \int dp \tilde{\Psi}_B(p) = \lambda \Psi_B(0) \frac{1}{2\pi} \int dp \frac{1}{E_B - \sqrt{p^2 + m^2}} \Rightarrow \\ \frac{1}{\lambda} &= \frac{1}{2\pi} \int dp \frac{1}{E_B - \sqrt{p^2 + m^2}}, \end{aligned} \quad (2.5)$$

which determines the bound state energy  $E_B$ . The resulting integral is logarithmically ultra-violet divergent and must hence be regularized. We do this by using dimensional regularization, i.e. by analytically continuing the spatial dimension to  $D = 1 + \varepsilon \in \mathbb{C}$  and by finally taking the limit  $\varepsilon \rightarrow 0$ . While the coupling constant  $\lambda$  is dimensionless in one dimension, in  $D$  dimensions the prefactor of the  $\delta$ -function has dimension  $(\text{mass})^{1-D}$ . In order to renormalize the bare coupling, we let it depend on the cut-off, and we replace  $\lambda$  by  $\lambda(\varepsilon)m^{-\varepsilon}$ . In order to keep  $\lambda(\varepsilon)$  dimensionless, we have factored out the dimensionful term  $m^{-\varepsilon} = m^{1-D}$ , using the particle mass  $m$  as the renormalization scale. The regularized gap equation then takes the form

$$\frac{m^{D-1}}{\lambda(\varepsilon)} = \frac{1}{(2\pi)^D} \int d^D p \frac{1}{E_B - \sqrt{p^2 + m^2}} = I(E_B). \quad (2.6)$$

For a bound state  $E_B < m$ , and we expand the integrand in powers of  $E_B/\sqrt{p^2 + m^2}$ , such that

$$I(E_B) = -\frac{\pi^{D/2} D}{\Gamma(D/2 + 1)} \int_0^\infty dp \frac{p^{D-1}}{\sqrt{p^2 + m^2}} \sum_{n=0}^\infty \left( \frac{E_B}{\sqrt{p^2 + m^2}} \right)^n. \quad (2.7)$$

While all higher-order terms are finite, the leading term (with  $n = 0$ ) is logarithmically ultra-violet divergent. All terms can be integrated separately, and then be re-summed, which in the limit  $\varepsilon \rightarrow 0$  yields

$$I(E_B) = m^\varepsilon \left[ \frac{1}{\pi\varepsilon} + \frac{\gamma - \log(4\pi)}{2\pi} - \frac{E_B}{2\pi\sqrt{m^2 - E_B^2}} \left( \pi + 2 \arcsin \frac{E_B}{m} \right) \right]. \quad (2.8)$$

Here  $\gamma \approx 0.5772$  is Euler's constant. For an ultra-strong bound state with energy  $E_B < -m$  the series from above diverges. Still, the result can be obtained by directly integrating the convergent expression

$$\begin{aligned} &\frac{1}{2\pi} \int dp \left( \frac{1}{E_B - \sqrt{p^2 + m^2}} + \frac{1}{\sqrt{p^2 + m^2}} \right) = \\ &\frac{E_B}{\pi\sqrt{E_B^2 - m^2}} \operatorname{arctanh} \frac{\sqrt{E_B^2 - m^2}}{E_B}. \end{aligned} \quad (2.9)$$

As a renormalization condition, we now hold the binding energy  $E_B$  fixed in units of the mass  $m$ , such that the running bare coupling is given by

$$\frac{1}{\lambda(\varepsilon)} = \frac{1}{\pi\varepsilon} + \frac{\gamma - \log(4\pi)}{2\pi} - \frac{E_B}{2\pi\sqrt{m^2 - E_B^2}} \left( \pi + 2 \arcsin \frac{E_B}{m} \right). \quad (2.10)$$

Let us consider the non-relativistic limit, in which the binding energy  $\Delta E_B = E_B - m$  is small compared to the rest mass. In that case, the running bare coupling is given by

$$\begin{aligned} \frac{1}{\lambda(\varepsilon)} &= \frac{1}{\pi\varepsilon} + \frac{\gamma - \log(4\pi)}{2\pi} - \frac{E_B}{2\pi\sqrt{m^2 - E_B^2}} \left( \pi + 2 \arcsin \frac{E_B}{m} \right) \\ &\rightarrow \frac{1}{\pi\varepsilon} + \frac{\gamma - \log(4\pi)}{2\pi} - \sqrt{\frac{m}{-2\Delta E_B}}. \end{aligned} \quad (2.11)$$

Interestingly, for the non-relativistic contact interaction  $\lambda\delta(x)$ , which does not require renormalization, for  $\lambda < 0$  the bound state energy is given by  $\Delta E_B = -m\lambda^2/2$  such that  $1/\lambda = -\sqrt{-m/2\Delta E_B}$ . This suggests to define a renormalized coupling

$$\frac{1}{\lambda(E_B)} = -\frac{E_B}{2\pi\sqrt{m^2 - E_B^2}} \left( \pi + 2 \arcsin \frac{E_B}{m} \right) < 0, \quad (2.12)$$

which is defined at the scale  $E_B$ . Dropping the terms  $1/\pi\varepsilon + [\gamma - \log(4\pi)]/2\pi$  corresponds to the modified minimal subtraction  $\overline{MS}$  scheme that is commonly used in quantum field theory.

Let us now determine the bound state wave function in coordinate space

$$\Psi_B(x) = \frac{A}{2\pi} \int dp \frac{\exp(ipx)}{E_B - \sqrt{p^2 + m^2}}. \quad (2.13)$$

The integration can be extended to the closed contour  $\Gamma$  illustrated in Figure 1. For  $0 < E_B < m$ , the integrand has a pole at  $p = i\sqrt{m^2 - E_B^2}$ , which is enclosed by  $\Gamma$ , as well as a branch cut along the positive imaginary axis starting at  $p = im$ . The wave function then takes the form

$$\Psi_B(x) = A \left[ \frac{1}{\pi} \int_m^\infty d\mu \frac{\sqrt{\mu^2 - m^2}}{E_B^2 - m^2 + \mu^2} \exp(-\mu|x|) + \frac{E_B \exp(-\sqrt{m^2 - E_B^2}|x|)}{\sqrt{m^2 - E_B^2}} \right]. \quad (2.14)$$

The integral results from the two contributions along the branch cut, while the last term is the residue of the pole at  $p = i\sqrt{m^2 - E_B^2}$ . As illustrated in Fig.2, the wave function is logarithmically divergent at the origin. This short-distance divergence is unaffected by the renormalization. In particular, the singularity of the wave function is integrable and it is thus normalizable in the usual sense. Alternatively, the bound

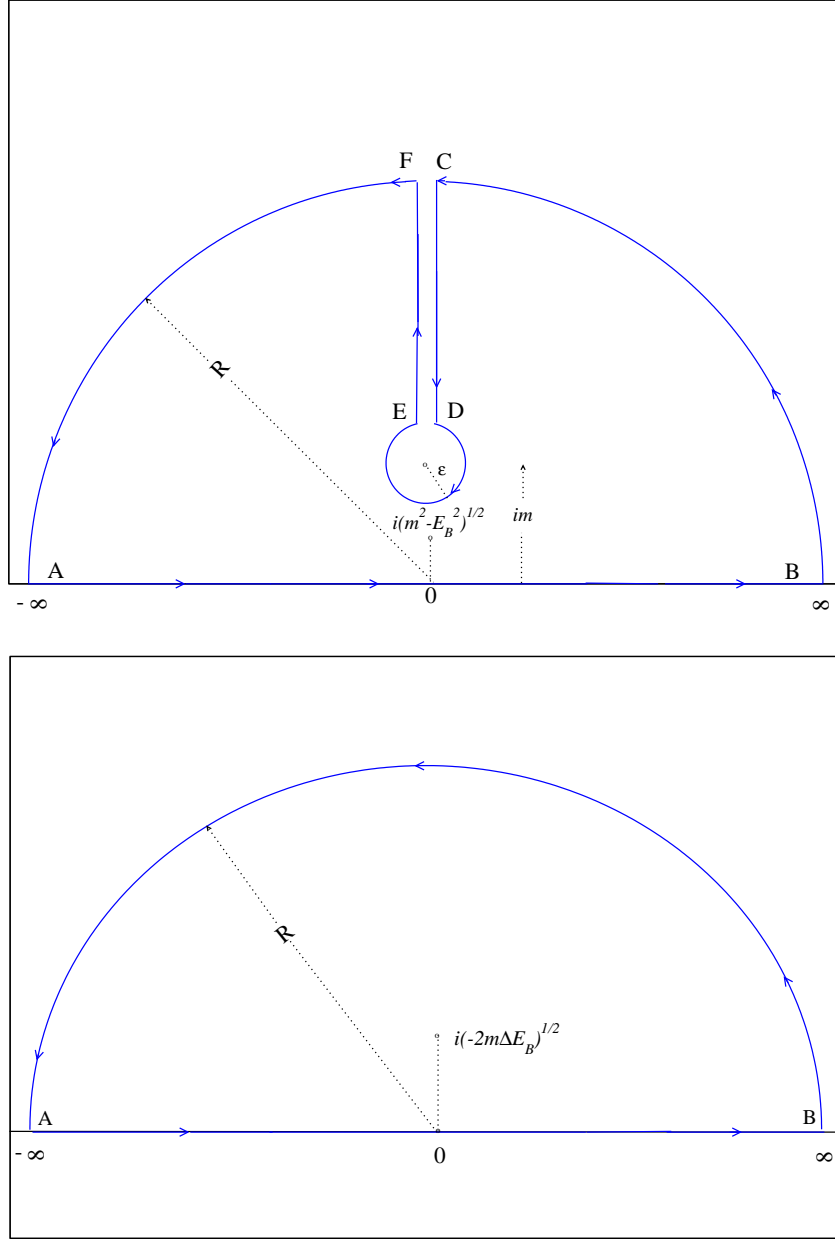


Figure 1: *Integration contours for the determination of the wave function of the bound state. In the relativistic case (top panel), there is a branch cut along the positive imaginary axis, starting at  $p = im$ . In addition, for  $0 < E_B < m$ , there is a pole at  $p = i\sqrt{m^2 - E_B^2}$ . In the non-relativistic case (bottom panel), there is still a pole, but no branch cut.*

state wave function can be expressed in terms of Bessel functions

$$\Psi_B(x) = \frac{A}{\sqrt{\pi}} \sum_{n=0}^{\infty} \left(\frac{E_B}{m}\right)^n \left(\frac{m|x|}{2}\right)^{n/2} \frac{K_{n/2}(m|x|)}{\Gamma\left(\frac{n+1}{2}\right)}. \quad (2.15)$$



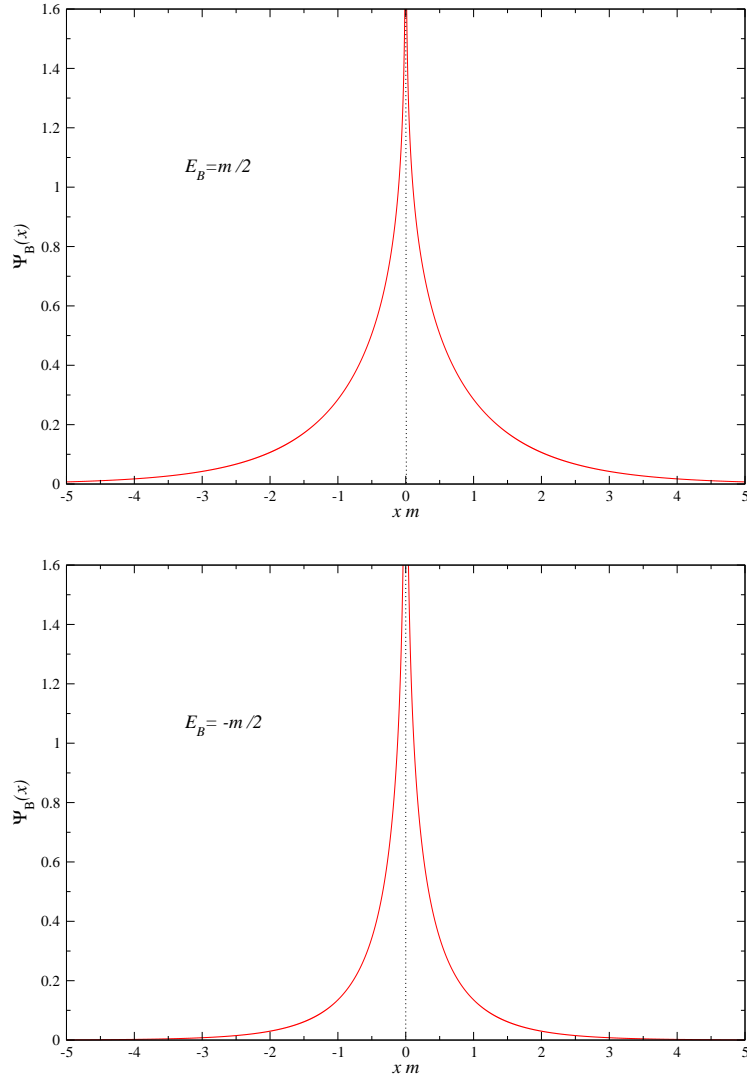


Figure 2: *Bound state wave function in coordinate space for an ordinary bound state with  $E_b = m/2$  (top panel), and for a strong bound state with  $E_b = -m/2$  (bottom panel).*

The normalization constant is most easily determined in momentum space

$$\begin{aligned} \frac{|A|^2}{2\pi} \int dp \frac{1}{(E_B - \sqrt{p^2 + m^2})^2} &= 1 \Rightarrow \\ \frac{2\pi}{|A|^2} &= \frac{2E_B}{m^2 - E_B^2} + \frac{m^2}{(m^2 - E_B^2)^{3/2}} \left( \pi + 2 \arcsin \frac{E_B}{m} \right). \end{aligned} \quad (2.16)$$

For the non-relativistic  $\delta$ -function potential, the wave function is finite at the origin and given by

$$\Psi_B(x) = \sqrt{\varkappa} \exp(-\varkappa|x|), \quad \Delta E_B = -\frac{\varkappa^2}{2m}. \quad (2.17)$$

In the non-relativistic limit, the relativistic wave function of eq.(2.14) reduces to

$$\Psi_B(x) = \sqrt{\varkappa} \left[ \frac{\varkappa}{m\pi} \int_m^\infty d\mu \frac{\sqrt{\mu^2 - m^2}}{\mu^2 - \varkappa^2} \exp(-\mu|x|) + \exp(-\varkappa|x|) \right]. \quad (2.18)$$

Since  $\varkappa/m \rightarrow 0$  in the non-relativistic limit, it indeed reduces to the non-relativistic wave function of eq.(2.17). However, the divergence of the relativistic wave function persists for any non-zero value of  $\varkappa/m$ . As we discussed in the introduction, a non-relativistic contact interaction is characterized by a 4-parameter family of self-adjoint extensions, while in the relativistic case there is only a 1-parameter family (parametrized by  $\lambda$ ). The other non-relativistic contact interactions can not be obtained by taking the non-relativistic limit of a relativistic theory.

Finally, let us also consider the strong bound states, for which the bound state energy  $E_B < 0$ , i.e. the binding energy  $\Delta E_B = E_B - m$  even exceeds the rest mass. In that case, the “pole” at  $p = i\sqrt{m^2 - E_B^2}$  has a vanishing residue and hence does not contribute to the result. The wave function of a strong bound state then takes the form

$$\Psi_B(x) = \frac{A}{\pi} \int_m^\infty d\mu \frac{\sqrt{\mu^2 - m^2}}{E_B^2 - m^2 + \mu^2} \exp(-\mu|x|), \quad E_B < 0. \quad (2.19)$$

The wave functions for a bound state with  $E_B = m/2$  and for a strong bound state with energy  $E_B = -m/2$  are illustrated in Figure 2. One would think that a relativistic system should not have negative total energy. In fact, the total energy should at least be as large as the positive rest mass of the system. In our case, translation invariance is explicitly broken by the contact potential, which means that the previous argument is not applicable here. One may think of the contact interaction as being generated by an infinitely heavy second particle located at  $x = 0$ . When the infinite mass of this particle is included in the total energy, it is indeed positive.

### 3 The Relativistic Probability Current

In the non-relativistic Schrödinger equation, probability is conserved because of the continuity equation

$$\partial_t \rho(x, t) + \partial_x j(x, t) = 0, \quad (3.1)$$

which relates the probability density  $\rho(x, t) = |\Psi(x, t)|^2$  to the probability current density  $j(x, t) = \frac{1}{2mi} [\Psi(x, t)^* \partial_x \Psi(x, t) - \partial_x \Psi(x, t)^* \Psi(x, t)]$ . While in the Dirac and Klein-Gordon equations, probability conservation is violated due to the presence of anti-particles, in the relativistic Schrödinger equation discussed here, there are no anti-particles and the continuity equation (3.1) still holds with the usual probability density  $\rho(x, t) = |\Psi(x, t)|^2$ , however, with the modified relativistic probability

current density, whose leading terms are

$$\begin{aligned}
j(x, t) &= \frac{1}{2mi} [\Psi(x, t)^* \partial_x \Psi(x, t) - \partial_x \Psi(x, t)^* \Psi(x, t)] \\
&+ \frac{1}{8m^3 i} [\Psi(x, t)^* \partial_x^3 \Psi(x, t) - \partial_x \Psi(x, t)^* \partial_x^2 \Psi(x, t) \\
&+ \partial_x^2 \Psi(x, t)^* \partial_x \Psi(x, t) - \partial_x^3 \Psi(x, t)^* \Psi(x, t)] + \dots
\end{aligned} \tag{3.2}$$

In momentum space the divergence  $\partial_x j(x, t)$  takes the compact form

$$p \tilde{j}(p, t) = \frac{1}{2\pi} \int dq \tilde{\Psi}(-q, t)^* [\sqrt{(p-q)^2 + m^2} - \sqrt{q^2 + m^2}] \tilde{\Psi}(p-q, t). \tag{3.3}$$

This expression trivially generalizes to an arbitrary energy-momentum dispersion relation  $E(p)$  and yields

$$\tilde{j}(p, t) = \frac{1}{2\pi} \int dq \tilde{\Psi}(-q, t)^* \frac{1}{p} [E(p-q) - E(q)] \tilde{\Psi}(p-q, t). \tag{3.4}$$

For a general dispersion relation, the bound state wave function in momentum space takes the form

$$\tilde{\Psi}_B(p) = \frac{A}{E_B - E(p)}. \tag{3.5}$$

The divergence of the probability density then automatically vanishes because

$$\begin{aligned}
p \tilde{j}(p) &= \frac{|A|^2}{2\pi} \int dq \frac{1}{E_B - E(q)} [E(p-q) - E(q)] \frac{1}{E_B - E(p-q)} \\
&= \frac{|A|^2}{2\pi} \int dq \left( \frac{1}{E_B - E(p-q)} - \frac{1}{E_B - E(q)} \right) = 0.
\end{aligned} \tag{3.6}$$

## 4 Dimensional Regularization and Renormalization of Scattering States

Let us now consider the scattering states. First of all, the states of odd parity, which vanish at the origin, are unaffected by the  $\delta$ -function potential. Hence, we limit ourselves to stationary scattering states of even parity, which we parametrize as

$$\tilde{\Psi}_E(p) = \delta(p - \sqrt{E^2 - m^2}) + \delta(p + \sqrt{E^2 - m^2}) + \tilde{\Phi}_E(p). \tag{4.1}$$

Later we will combine scattering states of even and odd parity in order to extract the reflection and transmission amplitudes. Inserting the ansatz from above in eq.(2.3), we obtain

$$\begin{aligned}
(\sqrt{p^2 + m^2} - E) \tilde{\Phi}_E(p) + \frac{\lambda}{\pi} + \frac{\lambda}{2\pi} \int dp' \tilde{\Phi}_E(p') &= 0 \Rightarrow \\
\tilde{\Phi}_E(p) &= \frac{\lambda}{\pi} \frac{1 + \pi \Phi_E(0)}{E - \sqrt{p^2 + m^2}}.
\end{aligned} \tag{4.2}$$

Integrating eq.(4.2) over all momenta one finds

$$\Phi_E(0) = \frac{1}{2\pi} \int dp \tilde{\Phi}_E(p) = \frac{\lambda}{\pi} [1 + \pi \Phi_E(0)] \frac{1}{2\pi} \int dp \frac{1}{E - \sqrt{p^2 + m^2}}. \quad (4.3)$$

Again, by replacing  $\lambda$  with  $\lambda(\varepsilon)m^{-\varepsilon}$ , and by using dimensional regularization, we then obtain

$$\Phi_E(0) = \frac{1}{\pi} \frac{\lambda(\varepsilon)m^{-\varepsilon}I(E)}{1 - \lambda(\varepsilon)m^{-\varepsilon}I(E)}. \quad (4.4)$$

For positive energy  $E$  the integral takes the form

$$I(E) = m^\varepsilon \left[ \frac{1}{\pi\varepsilon} + \frac{\gamma - \log(4\pi)}{2\pi} + \frac{E}{\pi\sqrt{E^2 - m^2}} \operatorname{arctanh} \frac{\sqrt{E^2 - m^2}}{E} \right]. \quad (4.5)$$

Using eq.(2.6), the function  $\tilde{\Phi}_E(p)$  then results as

$$\tilde{\Phi}_E(p) = \frac{\lambda(E, E_B)}{\pi} \frac{1}{E - \sqrt{p^2 + m^2}}, \quad (4.6)$$

with the energy-dependent running coupling constant (again renormalized at the scale  $E_B$ ) given by

$$\begin{aligned} \lambda(E, E_B) = \frac{1}{I(E_B) - I(E)} = & - \left[ \frac{E}{\pi\sqrt{E^2 - m^2}} \operatorname{arctanh} \frac{\sqrt{E^2 - m^2}}{E} \right. \\ & \left. + \frac{E_B}{2\pi\sqrt{m^2 - E_B^2}} \left( \pi + 2 \arcsin \frac{E_B}{m} \right) \right]^{-1}. \end{aligned} \quad (4.7)$$

Remarkably, using eqs.(2.8) and (4.5), the ultra-violet divergences of  $I(E)$  and  $I(E_B)$  cancel, such that the running coupling constant is finite when we take the limit  $\varepsilon \rightarrow 0$ .

In order to investigate whether the resulting system is self-adjoint, let us now check the orthogonality of the various states. First, we calculate the scalar product of the bound state and the scattering states

$$\begin{aligned} \langle \Psi_B | \Psi_E \rangle &= \frac{1}{2\pi} \int dp \frac{1}{E_B - \sqrt{p^2 + m^2}} \\ &\times [\delta(p - \sqrt{E^2 - m^2}) + \delta(p + \sqrt{E^2 - m^2}) + \tilde{\Phi}_E(p)] \\ &= \frac{1}{\pi(E_B - E)} \\ &+ \frac{1}{\pi(I(E_B) - I(E))} \frac{1}{2\pi} \int dp \frac{1}{E_B - \sqrt{p^2 + m^2}} \frac{1}{E - \sqrt{p^2 + m^2}}. \end{aligned} \quad (4.8)$$

The integral results in

$$\begin{aligned} & \frac{1}{2\pi} \int dp \frac{1}{E_B - \sqrt{p^2 + m^2}} \frac{1}{E - \sqrt{p^2 + m^2}} = \\ & \frac{1}{E - E_B} \frac{1}{2\pi} \int dp \left( \frac{1}{E_B - \sqrt{p^2 + m^2}} - \frac{1}{E - \sqrt{p^2 + m^2}} \right) = \frac{I(E_B) - I(E)}{E - E_B}, \end{aligned} \quad (4.9)$$

such that indeed  $\langle \Psi_B | \Psi_E \rangle = 0$ . This is also the case for a strong bound state with  $E_B < 0$ , and even for an ultra-strong bound state with  $E_B < -m$ . Next we investigate the orthogonality of the scattering states

$$\begin{aligned} \langle \Psi_{E'} | \Psi_E \rangle &= \frac{1}{\pi} \delta(\sqrt{E^2 - m^2} - \sqrt{E'^2 - m^2}) + \frac{\lambda(E, E_B)}{\pi^2(E - E')} + \frac{\lambda(E', E_B)}{\pi^2(E' - E)} \\ &+ \frac{\lambda(E, E_B)\lambda(E', E_B)}{\pi^2} \frac{1}{2\pi} \int dp \frac{1}{E - \sqrt{p^2 + m^2}} \frac{1}{E' - \sqrt{p^2 + m^2}} \\ &= \frac{1}{\pi} \delta(k - k') + \frac{1}{\pi^2(E - E')} \left[ \frac{1}{I(E_B) - I(E)} - \frac{1}{I(E_B) - I(E')} \right] \\ &+ \frac{1}{\pi^2} \frac{1}{I(E_B) - I(E)} \frac{1}{I(E_B) - I(E')} \frac{I(E') - I(E)}{E - E'} \\ &= \frac{1}{\pi} \delta(k - k'). \end{aligned} \quad (4.10)$$

Here we have introduced  $k = \sqrt{E^2 - m^2}$  and  $k' = \sqrt{E'^2 - m^2}$ . The orthogonality of the various states shows explicitly that, after regularization and renormalization, the resulting Hamiltonian is indeed self-adjoint.

The contour for the determination of the scattering wave function in coordinate space is illustrated in Figure 3. In addition to the branch cut, there are two poles on the real axis at  $p = \pm k = \pm \sqrt{E^2 - m^2}$ , which give rise to in- and out-going plane waves. When these poles are avoided by the contour, one obtains the contribution  $\tilde{\Phi}_E(x)$  to the total scattering wave function. The even-parity stationary scattering wave function in coordinate space

$$\begin{aligned} \Psi_E(x) &= A(k) \left[ \cos(kx) + \lambda(E, E_B) \frac{\sqrt{k^2 + m^2}}{k} \sin(k|x|) \right. \\ &\quad \left. - \frac{\lambda(E, E_B)}{\pi} \int_m^\infty d\mu \frac{\sqrt{\mu^2 - m^2}}{\mu^2 + k^2} \exp(-\mu|x|) \right], \quad E = \sqrt{k^2 + m^2}, \end{aligned} \quad (4.11)$$

is illustrated in Figure 4. Like the bound state wave function, it is logarithmically divergent at the origin.

Let us again consider the non-relativistic limit by considering small scattering energies, such that  $\Delta E = E - m \ll m$ , while also maintaining a small bound state

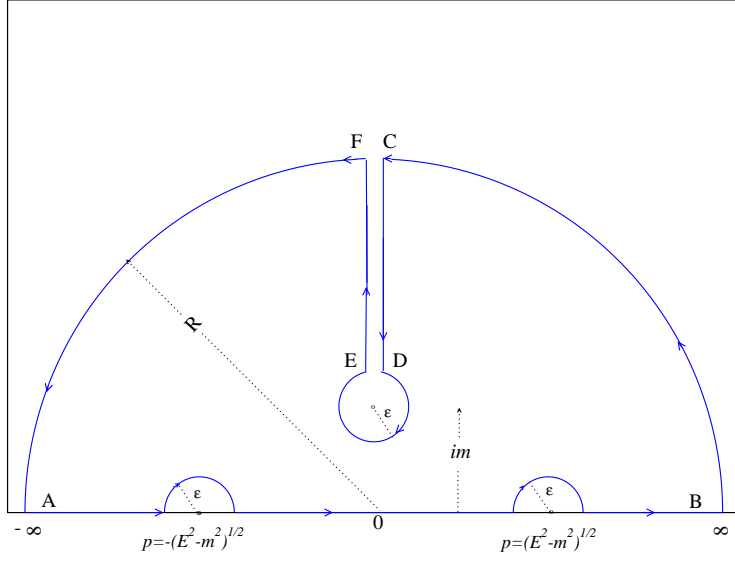


Figure 3: *Integration contour for the determination of the wave function of the scattering states with  $E > m$ . There is a branch cut along the positive imaginary axis, starting at  $p = im$ . In addition, there are two poles on the real axis at  $p = \pm\sqrt{E^2 - m^2}$ .*

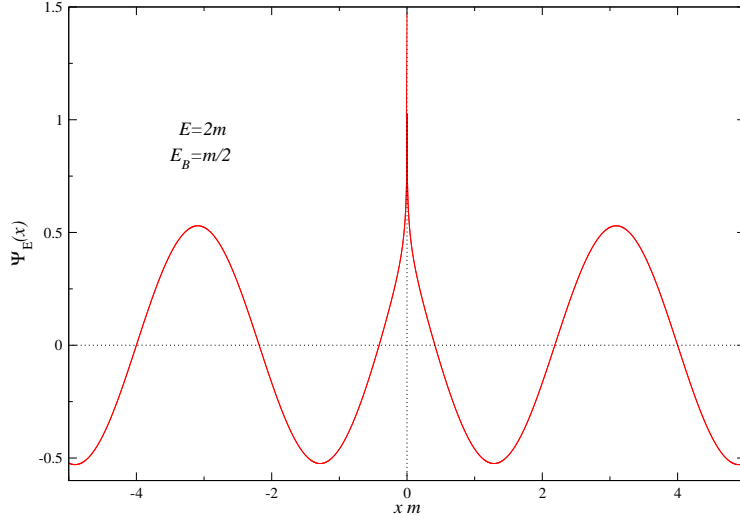


Figure 4: *Even-parity stationary scattering wave function in coordinate space for  $E = 2m$  and  $E_B = m/2$ . Like the wave function of the bound state, the scattering wave function also diverges logarithmically at the origin.*

energy  $|\Delta E_B| = |E_B - m| \ll m$ . In this case, the running coupling constant reduces to

$$\lambda(E, E_B) \rightarrow -\sqrt{-\frac{2\Delta E_B}{m}} = \lambda, \quad (4.12)$$

where  $\lambda$  is indeed the energy-independent coupling constant of the non-relativistic theory. As for the bound state wave function, the branch-cut contribution vanishes in the non-relativistic limit, such that one recovers the non-relativistic even-parity scattering wave function

$$\Psi_E(x) = A(k) \left[ \cos(kx) + \frac{\lambda m}{k} \sin(k|x|) \right]. \quad (4.13)$$

Here  $k = \sqrt{2m\Delta E}$ . It should be noted that the logarithmic divergence at the origin still persists for all non-zero values of  $\Delta E = k^2/2m$ .

## 5 Reflection and Transmission Amplitudes

Let us now construct reflection and transmission amplitudes by superimposing the non-trivial even-parity scattering states  $\Psi_E(x)$  with the trivial odd-parity scattering states  $B \sin(kx)$ . We will now adjust the amplitudes  $A(k)$  and  $B$  of the even and odd scattering states such that the wave function takes the form

$$\Psi_I(x) = \exp(ikx) + R(k) \exp(-ikx) + C(k)\lambda(E, E_B)\chi_E(x), \quad (5.1)$$

in region I to the left of the contact point, i.e. for  $x < 0$ . In region II, for  $x > 0$ , on the other hand, we demand

$$\Psi_{II}(x) = T(k) \exp(ikx) + C(k)\lambda(E, E_B)\chi_E(x). \quad (5.2)$$

Here

$$\chi_E(x) = \frac{1}{\pi} \int_m^\infty d\mu \frac{\sqrt{\mu^2 - m^2}}{\mu^2 + E^2 - m^2} \exp(-\mu|x|), \quad (5.3)$$

is the branch-cut contribution, which arises only in the relativistic case. Away from the contact point  $x = 0$ , this contribution decays exponentially and thus has no effect on the scattering wave function at asymptotic distances. After a straightforward calculation one obtains

$$A(k) = -C(k) = \frac{k}{k + i\sqrt{k^2 + m^2}\lambda(E, E_B)}, \quad B = i, \quad (5.4)$$

which leads to the reflection and transmission amplitudes

$$R(k) = -\frac{i\sqrt{k^2 + m^2}\lambda(E, E_B)}{k + i\sqrt{k^2 + m^2}\lambda(E, E_B)}, \quad T(k) = \frac{k}{k + i\sqrt{k^2 + m^2}\lambda(E, E_B)}, \quad (5.5)$$

which obey  $1 + R(k) = T(k)$ . Using eq.(4.7), it is straightforward to convince oneself that  $R(k)$  and  $T(k)$  have a pole at  $k = i\sqrt{m^2 - E_B^2}$ , which corresponds to the bound state with energy  $\sqrt{k^2 + m^2} = E_B$ . The S-matrix is given by

$$S(k) = R(k) + T(k) = \frac{k - i\sqrt{k^2 + m^2}\lambda(E, E_B)}{k + i\sqrt{k^2 + m^2}\lambda(E, E_B)} = \exp(2i\delta(k)), \quad (5.6)$$

which determines the scattering phase shift

$$\tan \delta(k) = -\frac{\sqrt{k^2 + m^2} \lambda(E, E_B)}{k}, \quad E = \sqrt{k^2 + m^2}. \quad (5.7)$$

In [42], the problem has been investigated using the self-adjoint extension theory of the pseudo-differential operator  $\sqrt{p^2 + m^2}$ , which led to the same expression for the S-matrix. This shows that dimensional regularization yields results that are consistent with the more abstract mathematical approach. We go significantly beyond the results of Ref.[42] by addressing numerous additional physics questions.

In three dimensions, it is common to consider the low-energy effective range expansion, which corresponds to  $k \cot \tilde{\delta}(k) = -1/a_0 + \frac{1}{2}r_0 k^2$ , where  $a_0$  is the scattering length and  $r_0$  is the effective range. The 3-d scattering phase shift  $\tilde{\delta}(k)$  measures the phase of the outgoing scattering wave relative to a sine-wave that vanishes at the origin. In our 1-d problem, there is no scattering in the odd-parity sine-wave channel. The non-trivial 1-d scattering phase  $\delta(k)$  measures the phase of the outgoing scattering wave relative to a cosine-wave that has a maximum at the origin. Hence, compared to the 3-d case,  $\delta(k)$  corresponds to  $\tilde{\delta}(k) + \frac{\pi}{2}$ , such that  $\cot \tilde{\delta}(k)$  corresponds to  $-\tan \delta(k)$ . Hence, in our 1-d case, the effective range expansion takes the form

$$-k \tan \delta(k) = -\frac{1}{a} + \frac{1}{2}r_0 k^2 + \dots \quad (5.8)$$

This yields the scattering length  $a_0$  and the effective range  $r_0$  as

$$a_0 = \frac{1}{m} \left( \frac{1}{\pi} - \frac{1}{\lambda(E_B)} \right), \quad r_0 = -\frac{1}{a_0 m^2} + \frac{2}{3\pi a_0^2 m^3}. \quad (5.9)$$

Here  $\lambda(E_B) < 0$  is the renormalized coupling constant defined in eq.(2.12). When there is a bound state, the scattering length is positive, and it diverges when the bound state approaches zero energy. In the absence of a bound state, the scattering lengths would become negative. The scale of  $r_0$  is set by the Compton wave length  $1/m$ , while its particular value is also influenced by the scattering length through the dimensionless combination  $am$ . The effective range vanishes in the non-relativistic limit  $am \rightarrow \infty$ , as one might naively expect for a contact interaction, but is non-zero in the relativistic case. This is due to the non-locality of the Hamiltonian  $\sqrt{p^2 + m^2}$ , which senses the contact interaction already from some distance  $r_0$ . The phase shift  $\delta(k)$  is illustrated in Figure 5. It varies between  $\delta(0) = \frac{\pi}{2}$  and  $\delta(\infty) = 0$ . This is consistent with the 1-d version of Levinson's theorem, which identifies the number of bound states as  $n = 2[\delta(0) - \delta(\infty)]/\pi$  [50, 51].

In the non-relativistic limit,  $\lambda(E, E_B)$  again reduces to the energy-independent coupling  $\lambda$  of the non-relativistic theory, such that we indeed recover the non-relativistic textbook results

$$R(k) = -\frac{im\lambda}{k + im\lambda}, \quad T(k) = \frac{k}{k + im\lambda}, \quad S(k) = \frac{k - im\lambda}{k + im\lambda}. \quad (5.10)$$



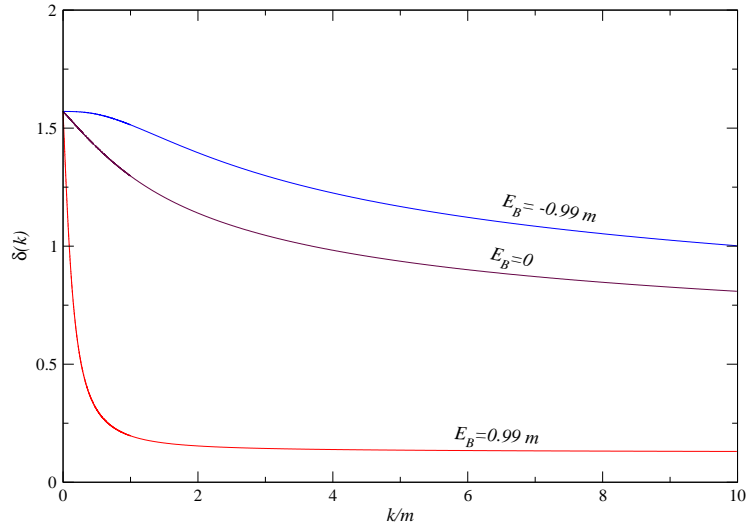


Figure 5: Phase shift  $\delta(k)$  as a function of the wave number  $k$ , for three different values of  $E_B/m = 0.99$ ,  $0$ , and  $-0.99$ .

These quantities have a pole at  $k = -im\lambda$ , which determines the non-relativistic bound state energy  $\Delta E_B = k^2/2m = -m\lambda^2/2$ . The scattering phase shift  $\delta(k)$  is then given by

$$\tan \delta(k) = -\frac{m\lambda}{k}, \quad (5.11)$$

which yields the scattering length  $a_0 = -1/(m\lambda)$  and the effective range  $r_0 = 0$ .

## 6 Running Coupling Constant, $\beta$ -Function, and Asymptotic Freedom

Until now, we have introduced the coupling  $\lambda(E_B)$  of eq.(2.12), which is renormalized at the bound state energy, as well as the energy-dependent running coupling  $\lambda(E, E_B)$  of eq.(4.7), which again uses  $E_B$  as the renormalization condition, and enters the reflection and transmission amplitudes in the same way as the energy-independent coupling  $\lambda$  in the non-relativistic case. Let us now investigate the dependence of the running coupling  $\lambda(E, E_B)$  on the scattering energy  $E$ , which is illustrated in Figure 6.

At high energies,  $\lambda(E, E_B)$  vanishes logarithmically, thus indicating that the scattered particle becomes free in the infinite energy limit. In particle physics, e.g. in QCD, this behavior is known as asymptotic freedom. The exact non-perturbative

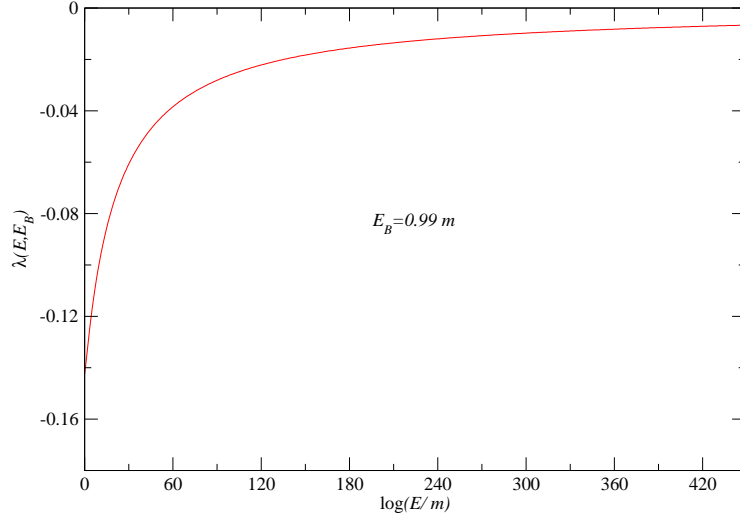


Figure 6: *Running coupling  $\lambda(E, E_B)$  as a function of the scattering energy  $E$ , for  $E_B = 0.99m$ .*

expression for the  $\beta$ -function takes the form

$$\begin{aligned} \beta(\lambda(E, E_B)) &= E \frac{\partial |\lambda(E, E_B)|}{\partial E} \\ &= -\frac{\lambda(E, E_B)^2}{\pi} + \frac{\lambda(E, E_B)^2 \epsilon^2}{1 - \epsilon^2} \left( \frac{1}{\lambda(E_B)} - \frac{1}{\lambda(E, E_B)} - \frac{1}{\pi} \right). \end{aligned} \quad (6.1)$$

Since  $\lambda(E, E_B)$  itself is negative, it is natural to use  $|\lambda(E, E_B)|$  to define the  $\beta$ -function. In the above expression,  $\epsilon = m/E \sim 2 \exp(-\pi/|\lambda(E, E_B)|)$  is non-perturbative and exponentially suppressed for small  $\lambda(E, E_B)$ . This implies that, to all orders in perturbation theory, the  $\beta$ -function is given by its 1-loop expression  $-\lambda(E, E_B)^2/\pi$ . The factor  $1/\pi$  plays the role of the 1-loop coefficient  $\beta_0$ . Non-perturbative corrections enter through  $\epsilon$ , and become noticeable only at low energies. For asymptotically large energies, the  $\beta$ -function behaves as

$$\beta(\lambda(E, E_B)) \rightarrow -\frac{\pi}{(\log(E/m))^2} \rightarrow -\frac{\lambda(E, E_B)^2}{\pi} < 0. \quad (6.2)$$

It vanishes at  $\lambda(E, E_B) \rightarrow 0$ , which corresponds to an ultra-violet fixed point. The negative sign of the  $\beta$ -function again signals asymptotic freedom. In Figure 7, the  $\beta$ -function is illustrated for different values of  $E_B/m$ , which influences the behavior only far away from the ultra-violet fixed point at  $\lambda(E, E_B) = 0$ .

Another zero of the  $\beta$ -function would require

$$\epsilon^2 \left( \frac{1}{\lambda(E_B)} - \frac{1}{\lambda(E, E_B)} \right) = \frac{1}{\pi} \Rightarrow \frac{E}{m} = \frac{m}{\sqrt{E^2 - m^2}} \operatorname{arctanh} \frac{\sqrt{E^2 - m^2}}{E}, \quad (6.3)$$

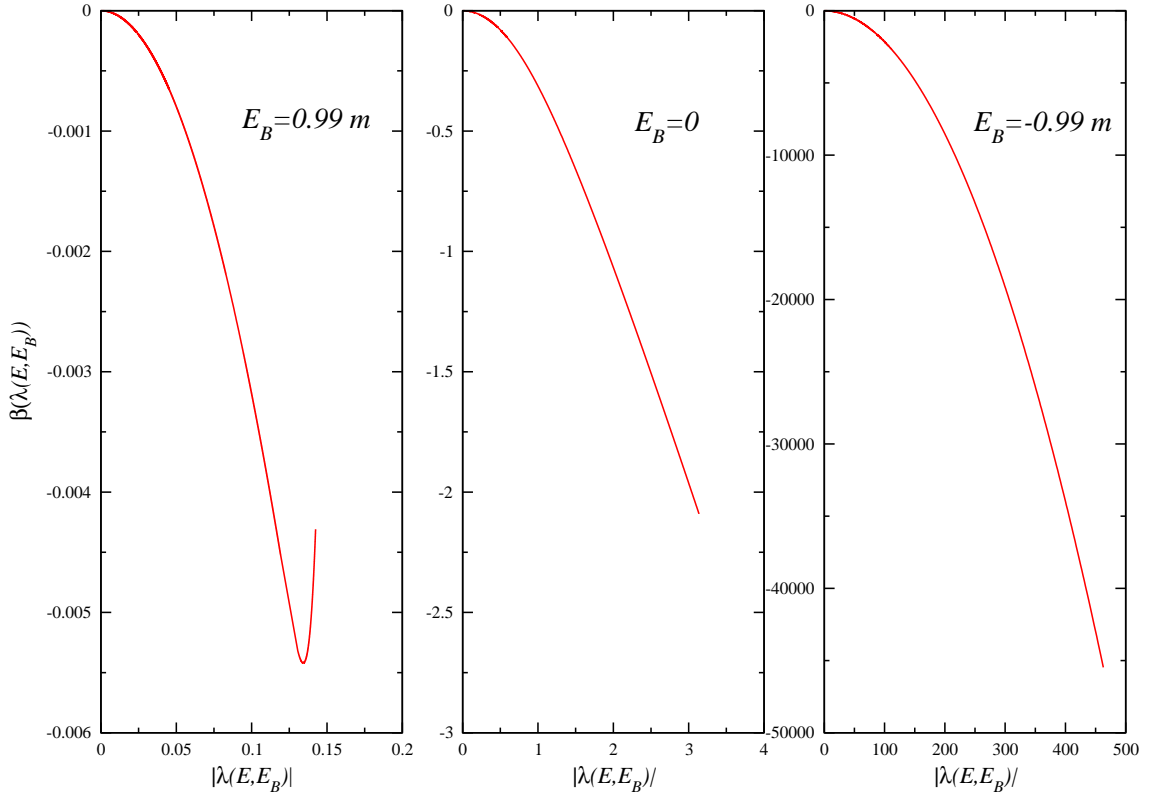


Figure 7: The  $\beta$ -function  $\beta(\lambda(E, E_B))$  as a function of the running coupling  $|\lambda(E, E_B)|$  for three values of  $E_B/m = 0.99, 0$ , and  $-0.99$ . The end points of the curves correspond to the maximal value of  $|\lambda(m, E_B)|$ , which is assumed in the low-energy limit  $E \rightarrow m$ . Note the different scales on the axes, which result from the very different ranges over which  $\lambda(E, E_B)$  is varying.

provided that  $\epsilon = m/E \neq 1$ . However, the above condition is satisfied only for  $E = m$ , and hence, in this case, no other fixed point exists. As we will see in Section 8, in the massless case,  $m = 0$ , there is an additional infra-red conformal fixed point.

## 7 Ultra-Strong Bound States and Repulsive Scattering States

Until now, we have used the expression of eq.(2.8) for  $I(E_B)$ , and thus we have implicitly assumed that  $|E_B| < m$ . This includes the case of strong bound states with  $-m < E_B < 0$ , but it excludes ultra-strong bound states with energies  $E_B < -m$ . We again point out that the strong and ultra-strong bound states are not necessarily tachyonic, because the  $\delta$ -function potential can be attributed to a hypothetical infinitely heavy particle. Hence, the total rest energy of the system always remains

positive. For  $E_B < -m$  one must use the expression of eq.(2.9) for  $I(E_B)$ , with interesting consequences for the bound and scattering state wave functions. First of all, it should be pointed out that the various states are still mutually orthogonal, such that the Hamiltonian remains self-adjoint, even in the presence of an ultra-strong bound state. This is easy to see, because the orthogonality relations eq.(4.8) and eq.(4.10) do not depend on the explicit form of  $I(E_B)$ .

Let us first consider the extreme limit  $E_B \rightarrow -\infty$ . In this case, the running coupling constant takes the form

$$\lambda(E, E_B) \rightarrow - \left[ \frac{E}{\pi \sqrt{E^2 - m^2}} \operatorname{arctanh} \frac{\sqrt{E^2 - m^2}}{E} - \frac{1}{\pi} \log \left( \frac{-2E_B}{m} \right) \right]^{-1}. \quad (7.1)$$

For small non-relativistic energies  $\Delta E = E - m \ll m$ , this reduces to

$$\lambda \rightarrow \frac{\pi}{\log(-2E_B/m)} > 0. \quad (7.2)$$

Remarkably, this  $\lambda$  actually plays the role of the strength of the repulsive contact interaction  $\lambda \delta(x)$  in the non-relativistic theory. In other words, despite the fact that there is an infinitely strongly bound state, the low-energy scattering states approach those of the non-relativistic repulsive potential  $\lambda \delta(x)$ , for which there is no bound state at all. In fact, in the limit  $E_B \rightarrow -\infty$ , the probability density of the relativistic ultra-strong bound state degenerates to a  $\delta$ -function. Because the scattering states still are logarithmically divergent at the origin, this is not in contradiction with orthogonality in the non-relativistic limit. Figure 8 compares the even-parity scattering wave functions at low energy in the relativistic and non-relativistic case, which indeed coincide, except in the ultimate vicinity of the contact point. This indeed makes sense, because the short-distance behavior of the relativistic and the non-relativistic theory are fundamentally different. We conclude that the relativistic contact interaction always produces a bound state. Remarkably, when this bound state becomes ultra-strong (with  $E_B \rightarrow -\infty$ ), it decouples from the scattering states, which behave as if the contact interaction was repulsive.

Let us now discuss the case  $-\infty < E_B < -m$ . In this case, the running coupling constant is given by

$$\begin{aligned} \lambda(E, E_B) = & - \left[ \frac{E}{\pi \sqrt{E^2 - m^2}} \operatorname{arctanh} \frac{\sqrt{E^2 - m^2}}{E} \right. \\ & \left. - \frac{E_B}{\pi \sqrt{E_B^2 - m^2}} \operatorname{arctanh} \frac{\sqrt{E_B^2 - m^2}}{E_B} \right]^{-1}. \end{aligned} \quad (7.3)$$

At low energies  $m < E < -E_B$ , the running coupling  $\lambda(E, E_B) > 0$  is repulsive, it diverges at  $E = -E_B$ , and becomes attractive (i.e.  $\lambda(E, E_B) < 0$ ) at high energies  $E > -E_B$ . The phase shift  $\delta(k)$  is illustrated in Figure 9. goes through a resonance at  $E = -E_B$  with  $\delta(\sqrt{E_B^2 - m^2}) = \frac{\pi}{2}$ . Since we still have  $\delta(0) = \frac{\pi}{2}$ , this behavior is still consistent with the 1-d version of Levinson's theorem.

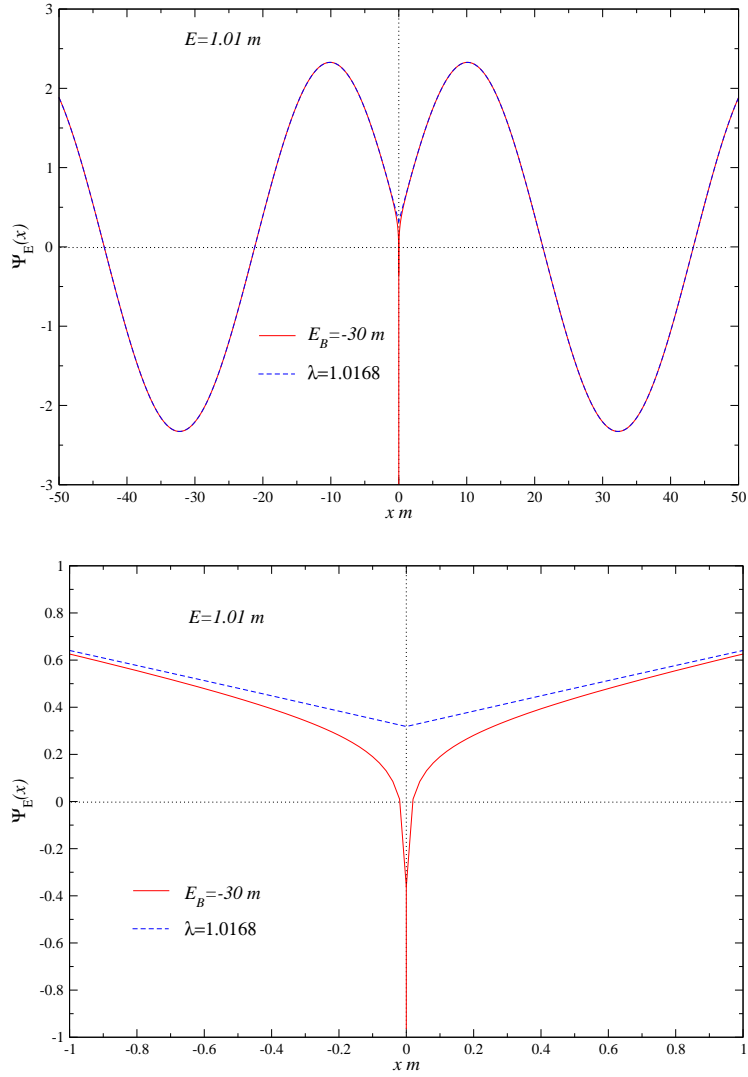


Figure 8: *Even-parity low-energy scattering wave function (with energy  $E = 1.01m$ ) in the presence of an ultra-strong bound state with  $E_B = -30m$ , compared to the corresponding non-relativistic wave function (with  $\lambda = 1.0168$ ). The panel on the bottom zooms into the region around the contact point  $x = 0$ , in which the relativistic wave function is logarithmically divergent, while the non-relativistic wave function remains finite.*

## 8 The Massless Case

Let us also consider the massless case  $m = 0$ . Since  $\lambda$  is dimensionless, the system is then scale-invariant, at least at the classical level. For  $m = 0$ , we are automatically limited to ultra-strong bound states (with  $E_B < -m$ ). The bound state energy  $E_B$  is a scale generated non-perturbatively at the quantum level, in a similar way as

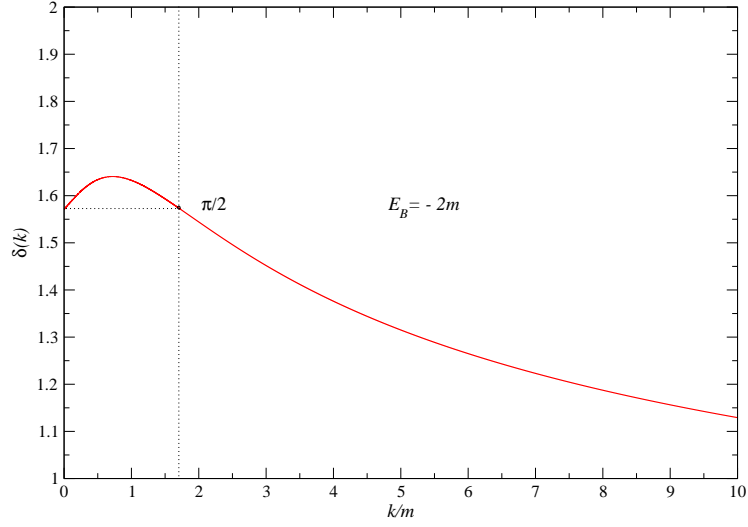


Figure 9: Phase shift  $\delta(k)$  as a function of the wave number  $k$  for  $E_B = -2m$ . The phase shift goes through a resonance at  $E = -E_B$  with  $\delta(k) = \delta(\sqrt{E_B^2 - m^2}) = \frac{\pi}{2}$ .

the proton mass is generated in massless QCD. Scale invariance is then anomalously broken and a scale, in this case  $E_B$ , emerges by dimensional transmutation. All physical quantities can then be expressed in units of this scale.

First, let us consider the bound state wave function in momentum space

$$\tilde{\Psi}_B(p) = \sqrt{\frac{\pi}{-E_B}} \frac{1}{E_B - |p|}. \quad (8.1)$$

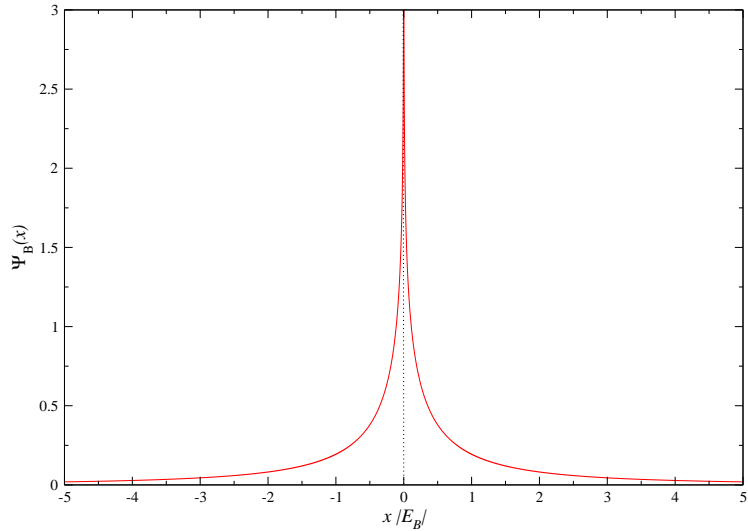


Figure 10: Bound state wave function in the massless case.

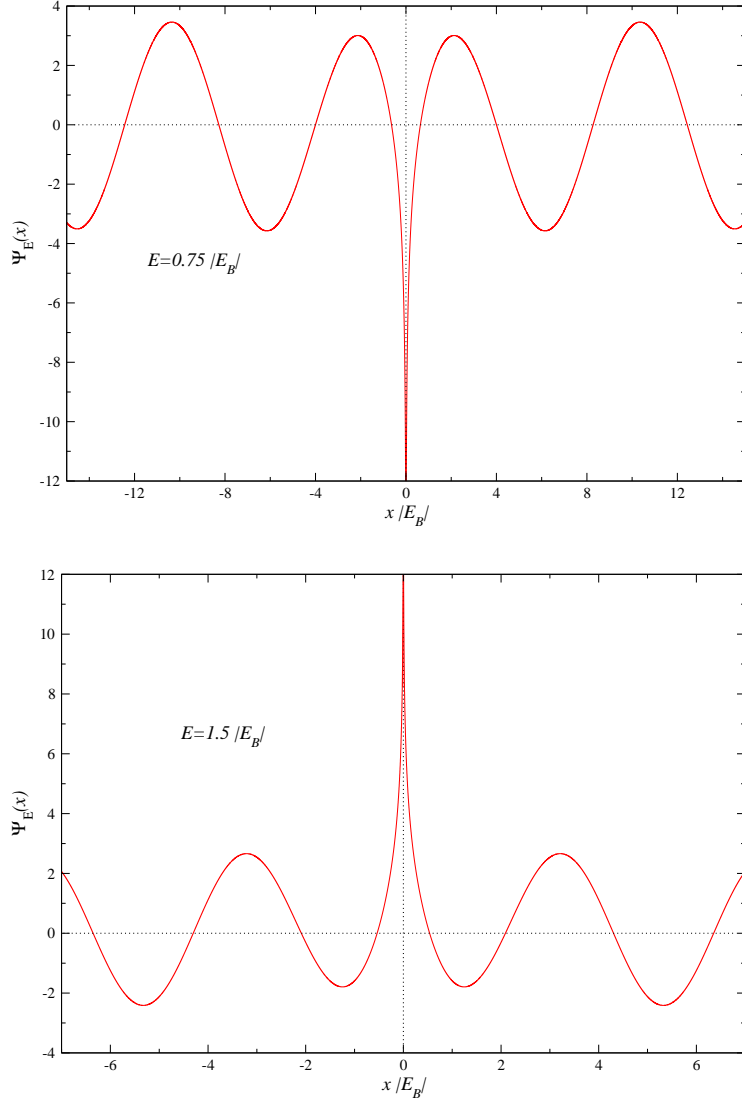


Figure 11: *Scattering wave functions in the massless case, for  $E = 0.75|E_B| < -E_B$  (top panel), and for  $E = 1.5|E_B| > -E_B$  (bottom panel).*

In coordinate space, it takes the form

$$\Psi_B(x) = \sqrt{\frac{1}{-\pi E_B}} \int_0^\infty d\mu \frac{\mu}{\mu^2 + E_B^2} \exp(-\mu|x|), \quad (8.2)$$

which is illustrated in Figure 10. As usual, the wave function diverges logarithmically at  $x = 0$ , but is still square-integrable. Next, we consider the even-parity scattering state (with  $E = k$ )

$$\begin{aligned} \Psi_E(x) &= A(k) [\cos(kx) + \lambda(E, E_B) \sin(k|x|)] \\ &- \frac{\lambda(E, E_B)}{\pi} \int_0^\infty d\mu \frac{\mu}{\mu^2 + k^2} \exp(-\mu|x|) \end{aligned} \quad (8.3)$$

Two scattering wave functions, one for  $m < E < -E_B$  and one for  $E > -E_B$ , are shown in Figure 11.

The resulting reflection and transmission amplitudes as well as the S-matrix are then given by

$$R(k) = -\frac{i\lambda(E, E_B)}{1 + i\lambda(E, E_B)}, \quad T(k) = \frac{1}{1 + i\lambda(E, E_B)}, \quad S(k) = \frac{1 - i\lambda(E, E_B)}{1 + i\lambda(E, E_B)}. \quad (8.4)$$

In the massless case, one obtains

$$\tan \delta(k) = -\lambda(E, E_B) = \frac{\pi}{\log(-E/E_B)} = \frac{\pi}{\log(k/|E_B|)}. \quad (8.5)$$

The  $\beta$ -function then reduces to

$$\beta(\lambda) = E \frac{\partial |\lambda(E, E_B)|}{\partial E} = -\frac{\pi}{(\log(-E/E_B))^2} = -\frac{\lambda(E, E_B)^2}{\pi}, \quad (8.6)$$

which is now valid even at low energies. The running coupling and the  $\beta$ -function are shown in Figure 12. Remarkably, the running coupling vanishes not only at high, but also at low energies. In fact, the theory has both an ultra-violet and an infra-red fixed point. At the ultra-violet fixed point,  $\lambda(E, E_B)$  approaches 0 from below, as  $E \rightarrow \infty$ , while at the infra-red fixed point,  $\lambda(E, E_B)$  approaches 0 from above, as  $E \rightarrow 0$ . Both fixed points are described by the same zero of the  $\beta$ -function of eq.(8.6).

This situation resembles the one of an asymptotically free non-Abelian gauge theory near the so-called conformal window, which is relevant in the context of walking technicolor theories [52–57]. Another system of this kind is the 2-d  $O(3)$  model at vacuum angle  $\theta = \pi$  [58, 59], whose low-energy effective theory is the conformal  $k = 1$  Wess-Zumino-Novikov-Witten model [60–62]. Such theories also have both an ultra-violet and an infra-red fixed point. While the theory is scale invariant at very low energies, scale invariance is still explicitly violated, via dimensional transmutation, at a non-perturbatively generated higher energy scale. Thanks to asymptotic freedom, this scale is exponentially small compared to the ultimate ultra-violet cut-off (which can thus be sent to infinity). In our model, the energy  $E_B < 0$  of the bound state sets the non-perturbatively generated energy scale, which still affects the scattering states at high energies  $E > -E_B$ . Low-energy scattering states (with  $0 < E \ll -E_B$ ) are governed by the infra-red fixed point and can thus be mapped into each other by scale transformations, as illustrated in Figure 13.

## 9 Conclusions

We have investigated contact interactions in 1-dimensional relativistic quantum mechanics. In contrast to the non-relativistic case, there is only a 1-parameter family



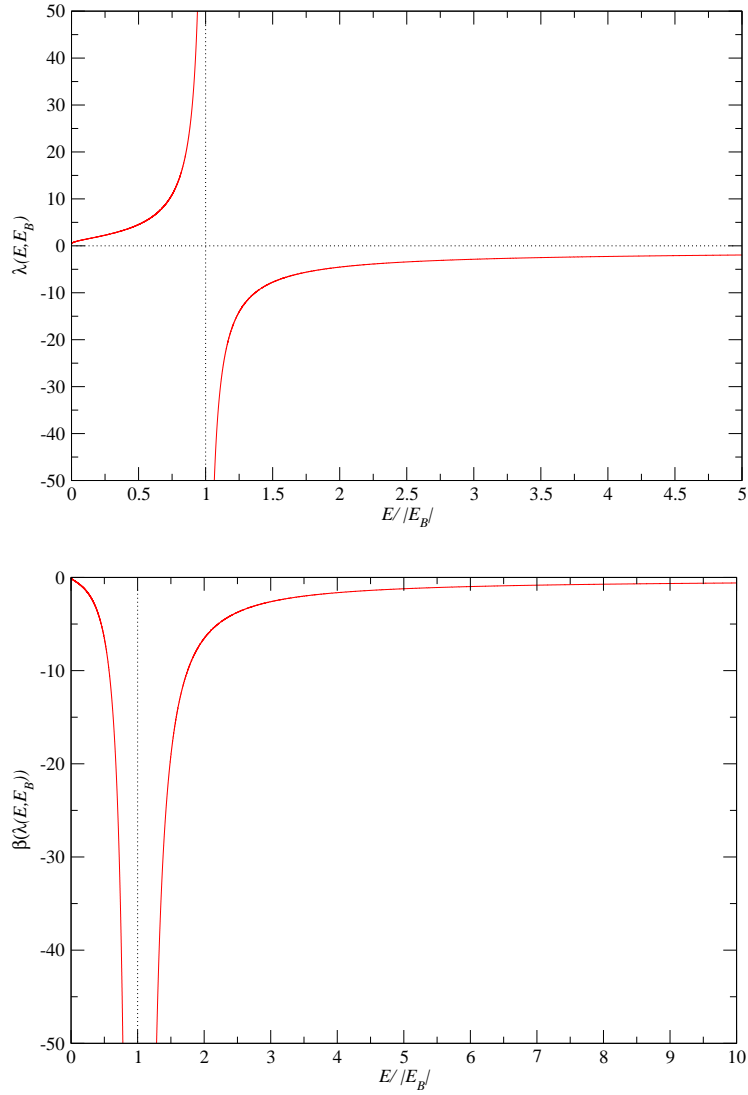


Figure 12: *Top panel: The running coupling  $\lambda(E, E_B)$  as a function of the energy  $E$  in the massless case. The coupling goes to zero both at high and at low energies. Bottom panel: The  $\beta$ -function  $\beta(\lambda(E, E_B))$  as a function of the scattering energy  $E$  (in units of  $|E_B|$ ) in the massless limit.*

of self-adjoint extensions of the pseudo-differential operator  $H = \sqrt{p^2 + m^2}$ , which is characterized by the contact potential  $\lambda\delta(x)$ . Remarkably, this simple potential gives rise to rather rich physics. First of all, unlike in the non-relativistic case, the  $\delta$ -function potential requires regularization and subsequent renormalization, which we have performed using dimensional regularization. Indeed, using this physics approach, we obtained results that are consistent with the more abstract mathematical theory of self-adjoint extensions of pseudo-differential operators. That theory also implies that there are no non-trivial relativistic contact interactions in more than

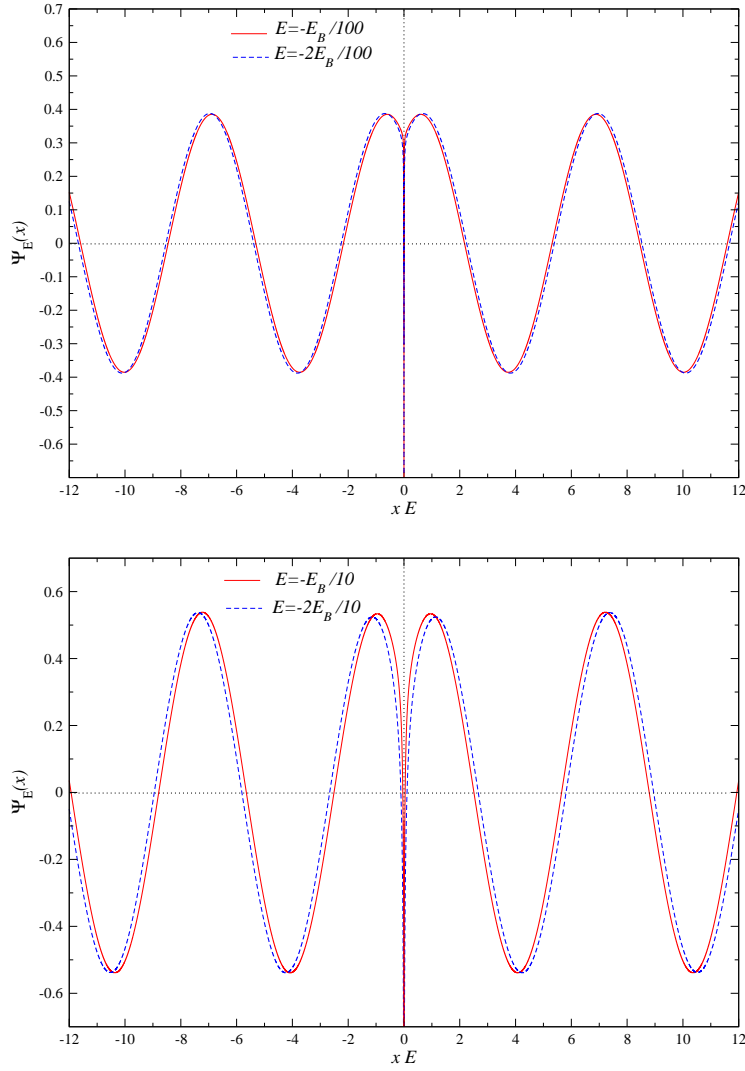


Figure 13: *Top panel: Even-parity scattering wave functions at low energies  $E = |E_B|/100$  and  $E = 2|E_B|/100$ , very close to the infra-red conformal fixed point, as a function of the rescaled position  $x E$ . The two wave functions are related by a factor 2 scale transformation. Bottom panel: The scattering wave functions at somewhat higher energies  $E = |E_B|/10$  and  $E = 2|E_B|/10$ , further away from the conformal fixed point, show a visible deviation from scale invariance.*

one spatial dimension. This is again in contrast to the non-relativistic case, in which there is a 1-parameter family of non-trivial contact interactions both in two and three spatial dimensions. In four and more spatial dimensions, on the other hand, there are no non-trivial self-adjoint extensions of the non-relativistic free particle Hamiltonian. It is interesting to investigate contact interactions in higher dimensions also using dimensional regularization. This has already been done in the non-relativistic case. While dimensional regularization provides results that are consistent with the

self-adjoint extension theory in two and three spatial dimensions, in contrast to the theory of self-adjoint extensions, it seems to lead to non-trivial contact interactions in higher dimensions [34]. However, it turns out that the resulting Hamiltonian is not self-adjoint and thus not physically meaningful. In this sense, dimensional regularization actually fails to produce the correct result. We suspect that the same may happen in the relativistic case, already in two and three spatial dimensions, which might be worth investigating.

As we discussed before, the external  $\delta$ -function potential can be attributed to an infinitely heavy particle. It is interesting to ask whether this second particle can be treated fully dynamically, by giving it a finite mass. Only then the system may become Poincaré invariant, because translation invariance is no longer explicitly broken by the position of the external contact interaction. Leutwyler’s non-interaction theorem suggests that Poincaré invariance is incompatible with interacting point particles. However, since the theorem operates at the classical level, and does not apply to quantum mechanical point interactions, there may be a quantum loop-hole that would be worth exploring. For the fully dynamical two-particle problem, the question arises whether both a self-adjoint Hamiltonian and a self-adjoint boost operator can be constructed, which obey the commutation relations of the Poincaré algebra together with the operator of the total momentum  $P$ . If so, the two-particle system will have a total energy  $E = \sqrt{P^2 + M^2}$ , where  $M$  is the rest-energy of the system. In such a system, one could also investigate the Lorentz contraction of a moving wave packet, which, until now, has been investigated for free particles only [14]. Although we know that Nature makes relativistic “particles” as non-local quantized field excitations, at least for pedagogical reasons, it is interesting to explore the alternative possibilities of local relativistic point particles. Based on the non-interaction theorem, such alternatives are expected to be very limited, which, in turn, underscores the strengths of relativistic quantum field theories.

As we have shown, asymptotic freedom, dimensional transmutation, and an infra-red conformal fixed point in the massless limit, already arise in 1-dimensional relativistic point particle quantum mechanics with a  $\delta$ -function potential. This allowed us to illustrate non-trivial quantum field theoretical phenomena as well as techniques including dimensional regularization and renormalization, avoiding the technical complications of quantum field theory. We conclude this paper by expressing our hope that the relatively simple system that we have investigated here will help to bridge the large gap that separates non-relativistic quantum mechanics from relativistic quantum field theory in the teaching of fundamental physics.

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